

On k -ended spanning and dominating trees

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Abstract

A tree with at most k leaves is called a k -ended tree. A spanning 2-ended tree is a Hamilton path. A Hamilton cycle can be considered as a spanning 1-ended tree. The earliest result concerning spanning trees with few leaves states that if k is a positive integer and G is a connected graph of order n with $d(x) + d(y) \geq n - k + 1$ for each pair of nonadjacent vertices x, y , then G has a spanning k -ended tree. In this paper, we improve this result in two ways, and an analogous result is proved for dominating k -ended trees based on the generalized parameter t_k - the order of a largest k -ended tree. In particular, t_1 is the circumference (the length of a longest cycle), and t_2 is the order of a longest path.

Key words. Hamilton cycle, Hamilton path, dominating cycle, dominating path, longest path, k -ended tree.

1 Introduction

Throughout this article we consider only finite undirected graphs without loops or multiple edges. The set of vertices of a graph G is denoted by $V(G)$ and the set of edges by $E(G)$. A good reference for any undefined terms is [1].

For a graph G , we use n , δ and α to denote the order (the number of vertices), the minimum degree and the independence number of G , respectively. For a subset $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S . If $\alpha \geq k$ for some integer k , let σ_k be the minimum degree sum of an independent set of k vertices; otherwise we let $\sigma_k = +\infty$.

If Q is a path or a cycle in a graph G , then the order of Q , denoted by $|Q|$, is $|V(Q)|$. Each vertex and edge in G can be interpreted as simple cycles of orders 1 and 2, respectively. The graph G is hamiltonian if G contains a Hamilton cycle, i.e. a cycle containing every vertex of G . A cycle C of G is said to be dominating if $V(G - C)$ is an independent set of vertices.

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We write a cycle Q with a given orientation by \vec{Q} . For $x, y \in V(Q)$, we denote by $x\vec{Q}y$ the subpath of Q in the chosen direction from x to y . For $x \in V(Q)$, we denote the successor and the predecessor of x on \vec{Q} by x^+ and x^- , respectively.

A vertex of degree one is called an end-vertex, and an end-vertex of a tree is usually called a leaf. The set of end-vertices of G is denoted by $End(G)$. For a positive integer k , a tree T is said to be a k -ended tree if $|End(T)| \leq k$. A Hamilton path is a spanning 2-ended tree. A Hamilton cycle can be interpreted as a spanning 1-ended tree. In particular, K_2 is hamiltonian and is a 1-ended tree. We denote by t_k the order of a largest k -ended tree in G . In particular, t_1 is the order of a longest cycle, and t_2 is the order of a longest path in G .

For two vertices u and v of G , let $d_G(u, v)$ denote the distance between u and v . For a vertex v of G , the distance between v and a subgraph H is defined to be the minimum value of $d_G(v, x)$ for all $x \in V(H)$, and denoted by $d_G(v, H)$. If $X \subseteq V(G)$ then $d_G(v, X)$ is defined analogously.

Our starting point is the earliest sufficient condition for a graph to be hamiltonian due to Dirac [3].

Theorem A [3]. Every graph with $\delta \geq \frac{n}{2}$ is hamiltonian.

In 1960, Ore [10] improved Theorem A by replacing the minimum degree δ with the arithmetic mean $\frac{1}{2}\sigma_2$ of two smallest degrees among pairwise nonadjacent vertices.

Theorem B [10]. Every graph with $\sigma_2 \geq n$ is hamiltonian.

The analog of Theorem B for Hamilton paths follows easily.

Theorem C [10]. Every graph with $\sigma_2 \geq n - 1$ has a Hamilton path.

In 1971, Las Vergnas [4] gave a degree condition that guarantees that any forest in G of limited size and with a limited number of leaves can be extended to a spanning tree of G with a limited number of leaves in an appropriate sense. This result implies as a corollary a degree sum condition for the existence of a tree with at most k leaves including Theorem B and Theorem C as special cases for $k = 1$ and $k = 2$, respectively.

Theorem D [2], [4], [7]. If G is a connected graph with $\sigma_2 \geq n - k + 1$ and k a positive integer, then G has a spanning k -ended tree.

However, Theorem D was first openly formulated and proved in 1976 by the author [7] and was reproved in 1998 by Broersma and Tuinstra [2]. Moreover, the full characterization of connected graphs without spanning k -ended trees was given in [6] when $\sigma_2 \geq n - k$ including well-known characterization of connected graphs without Hamilton cycles when $\sigma_2 \geq n - 1$. This particular result was

reproved in 1980 by Nara Chie [5].

In this paper we prove that the connectivity condition in Theorem D can be removed, and the conclusion can be strengthened.

Theorem 1. If G is a graph with $\sigma_2 \geq n - k + 1$ and k a positive integer, then G has a spanning k -ended forest.

The next improvement of Theorem D is based on parameter t_k including the circumference (the length of a longest cycle) and the length of a longest path in a graph for $k = 1$ and $k = 2$, respectively.

Theorem 2. Let G be a connected graph with $\sigma_2 \geq t_{k+1} - k + 1$ and k a positive integer. Then G has a spanning k -ended tree.

The graph $(\delta + k)K_1 + K_\delta$ shows that the bound $t_{k+1} - k + 1$ in Theorem 2 cannot be relaxed to $t_k - k + 1$.

Finally, we give a degree sum condition for dominating k -ended trees.

Theorem 3. If G is a connected graph with $\sigma_3 \geq t_{k+1} - 2k + 4$ for some integer $k \geq 2$, then G has a dominating k -ended tree.

The graph $(\delta + k - 1)K_2 + K_{\delta-1}$ shows that the bound $t_{k+1} - 2k + 4$ in Theorem 3 cannot be relaxed to $t_k - 2k + 4$.

The following corollary follows immediately.

Corollary 1. If G is a connected graph with $\sigma_3 \geq n - 2k + 4$ for some integer $k \geq 2$, then G has a dominating k -ended tree.

The graph $(\delta + k - 1)K_2 + K_{\delta-1}$ shows that the bound $\sigma_3 \geq t_{k+1} - 2k + 4$ in Theorem 3 cannot be relaxed to $\sigma_3 \geq t_{k+1} - 2k + 3$.

We present also some earlier results concerning spanning k -ended trees that are not included in the recent survey paper [11]. We call a graph G hypo- k -ended if G has no a spanning k -ended tree, but for any $v \in V(G)$, $G - v$ has a spanning k -ended tree.

Theorem E [8]. For each $k \geq 3$, the minimum number of vertices (edges, faces, respectively) of a simple 3-polytope without a spanning k -ended tree is $8 + 3k$ ($12 + 6k$, $6 + 3k$, respectively).

Theorem F [9]. For each $n \geq 17k$ and $k \geq 2$, except possible for $n = 17k + 1$, $17k + 2$, $17k + 4$ and $17k + 7$, there exist hypo- k -ended graphs of order n .

2 Proofs

Proof of Theorem 1. Let G be a graph with $\sigma_2 \geq n - k + 1$ and let H_1, \dots, H_m be the connected components of G . Let $\vec{P} = x\vec{P}y$ be a longest path in H_1 . If $|P| \geq n - k + 2$ then $|G - P| = n - |P| \leq k - 2$, implying that G has a spanning k -ended forest. Now let $|P| \leq n - k + 1$. Since P is extreme, we have $N(x) \cup N(y) \subseteq V(P)$. Recalling also that $\sigma_2 \geq n - k + 1$, we have (by standard arguments) $N(x) \cap N^+(y) \neq \emptyset$, implying that $G[V(P)]$ is hamiltonian. Further, if $|V(P)| < |V(H_1)|$ then we can form a path longer than P , contradicting the maximality of P . Hence, $|V(P)| = |V(H_1)|$, that is H_1 is hamiltonian as well. By a similar argument, H_i is hamiltonian for each $i \in \{1, \dots, m\}$ and therefore, has a spanning tree with exactly one leaf. Thus, G has a spanning forest with exactly m leaves.

It remains to show that $m \leq k$. If $m = 1$ then G has a spanning 1-ended tree and therefore, has a spanning k -ended tree. Let $m \geq 2$ and let $x_i \in V(H_i)$ ($i = 1, \dots, m$). Clearly, $\{x_1, x_2, \dots, x_m\}$ is an independent set of vertices. Since $d(x_i) \leq |V(H_i)| - 1$, we have

$$\sigma_2 \leq \sigma_m \leq \sum_{i=1}^m d(x_i) \leq \sum_{i=1}^m |V(H_i)| - m = n - m.$$

On the other hand, by the hypothesis, $\sigma_2 \geq n - k + 1$, implying that $m \leq k - 1$. ■

Proof of Theorem 2. Let G be a connected graph with $\sigma_2 \geq t_{k+1} - k + 1$ for some positive integer k .

Case 1. G is hamiltonian.

By the definition, G has a spanning 1-ended tree T_1 . Since $k \geq 1$, T_1 is a spanning k -ended tree.

Case 2. G is not hamiltonian.

Let T_2 be a longest path in G .

Case 2.1. $\sigma_2 \geq t_2$.

By standard arguments, $G[V(T_2)]$ is hamiltonian. If $t_2 < n$ then recalling that G is connected, we can form a path longer than T_2 , contradicting the maximality of T_2 . Otherwise G is hamiltonian and we can argue as in Case 1.

Case 2.2. $\sigma_2 \leq t_2 - 1$.

If $k = 1$ then by the hypothesis, $\sigma_2 \geq t_2$, implying that G is hamiltonian and we can argue as in Case 1. Let $k \geq 2$. Extend T_2 to a k -ended tree T_k and assume that T_k is as large as possible. If T_k is a spanning tree then we are done. Let T_k is not spanning. Then $|End(T_k)| = k$ since otherwise we can form a new k -ended tree larger than T_k , contradicting the maximality of T_k . Now extend T_k to a largest $(k + 1)$ -ended tree T_{k+1} . Recalling that T_k is a largest k -ended

tree, we get $|End(T_{k+1})| = k + 1$ and therefore,

$$t_{k+1} \geq |T_{k+1}| = |T_2| + |T_{k+1} - T_2|.$$

Observing that $|T_2| = t_2$ and $|T_{k+1} - T_2| \geq |End(T_{k+1})| - 2 = k - 1$, we get

$$t_{k+1} \geq t_2 + k - 1 \geq \sigma_2 + k,$$

contradicting the hypothesis. \blacksquare

Proof of Theorem 3. Let G be a connected graph with $\sigma_3 \geq t_{k+1} - 2k + 4$ for some integer $k \geq 2$, and let $\vec{T}_2 = x\vec{T}_2y$ be a longest path in G . If T_2 is a dominating path then we are done. Otherwise, since G is connected, we can choose a path $\vec{Q} = w\vec{Q}z$ such that $V(T_2 \cap Q) = \{w\}$ and $|Q| \geq 3$. Assume that $|Q|$ is as large as possible. Put $T_3 = T_2 \cup Q$. Since T_2 and Q are extreme, we have $N(x) \cup N(y) \subseteq V(T_2)$ and $N(z) \subseteq V(T_3)$. Let w^+ be the successor of w on T_2 . If $xy \in E$ then $T_3 + xy - w^+w$ is a path longer than T_2 , a contradiction. Let $xy \notin E$. By the same reason, we have $xz, yz \notin E$. Thus, $\{x, y, z\}$ is an independent set of vertices.

Claim 1. $N^-(x) \cap N^+(y) \cap N(z) = \emptyset$.

Proof. Assume the contrary.

Case 1. $v \in N^-(x) \cap N^+(y)$.

If $v = w$ then $xv^+ \in E$ and $T_3 + xv^+ - vv^+$ is a path longer than T_2 , a contradiction. Suppose without loss of generality that $v \in V(w^+\vec{T}_2y)$. If $v = w^+$ then $T_3 + xv^+ - wv - vv^+$ is a path longer than T_2 , a contradiction. Finally, if $v \in V(w^{+2}\vec{T}_2y)$ then

$$T_3 + xv^+ + yv^- - vv^- - vv^+ - ww^+$$

is a path longer than T_2 , a contradiction.

Case 2. $v \in N^-(x) \cap N(z)$.

If $v \in V(x\vec{T}_2w^{-2})$ then

$$T_2 + xv^+ + zv - vv^+ - ww^-$$

is a path longer than T_2 , a contradiction. Next, if $v = w^-$ then $T_2 + zw^- - ww^-$ is a path longer than T_2 , a contradiction. Further, if $v = w$ then $T_2 + xv^+ - ww^+$ is a path longer than T_2 , a contradiction. Finally, if $v \in V(w^+\vec{T}_2y)$ then

$$T_2 + xv^+ + zv - ww^+ - vv^+$$

is a path longer than T_2 , a contradiction.

Case 3. $v \in N^+(y) \cap N(z)$.

By a symmetric argument, we can argue as in Case 2. Claim 1 is proved. \triangle

By Claim 1,

$$\begin{aligned} t_3 \geq |T_3| &\geq |N^-(x)| + |N^+(y)| + |N(z)| + |\{z\}| \\ &= d(x) + d(y) + d(z) + 1 \geq \sigma_3 + 1. \end{aligned} \tag{1}$$

If $k = 2$ then by the hypothesis, $\sigma_3 \geq t_{k+1} - 2k + 4 = t_3$, contradicting (1). Let $k \geq 3$. If T_3 is a dominating 3-ended tree then clearly we are done. Otherwise $G - T_3$ contains an edge and we can extend T_3 to a largest 4-ended tree T_4 with $|T_4| \geq |T_3| + 2$. If $k = 3$, then by the hypothesis, $\sigma_3 \geq t_{k+1} - 2k + 4 = t_4 - 2$. On the other hand, by (1), $t_4 \geq |T_4| \geq |T_3| + 2 \geq \sigma_3 + 3$, a contradiction. Hence $k \geq 4$. If T_4 is dominating, then we are done. Otherwise we can extend T_4 to a largest 5-ended tree T_5 with $|T_5| \geq |T_4| + 2 \geq |T_3| + 4$. This procedure may be repeated until a dominating $(r + 1)$ -ended tree T_{r+1} is found. If $r + 1 \leq k$ then we are done. Let $r \geq k$. Then

$$\begin{aligned} t_{k+1} &\geq |T_{k+1}| \geq |T_3| + 2(k - 2) \\ &\geq \sigma_3 + 2k - 3 \geq t_{k+1} + 1, \end{aligned}$$

a contradiction. The proof is complete. \blacksquare

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